The effect of a strong magnetic field on two-dimensional flows of a conducting fluid

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The motion of a viscous, electrically conducting fluid past a finite two-dimensional obstacle is investigated. The magnetic field is assumed to be uniform and parallel to the velocity at infinity. By means of a perturbation technique, approximations valid for large values of the Hartmann number M are derived. It is found that, over any finite region, the flow field is characterized by the presence of shear layers fore and aft of the body. The limit attained over the exterior region represents the two-dimensional counterpart of the axially symmetric solution given by Chester (1961). Attention is focused on a number of nominally 'higher-order' effects, including the presence of two distinct boundary layers. The results hold only when $M \gg Re$; Re = Reynolds number. However, a generalization of the procedure, in which the last assumption is relaxed, is suggested.

1. Introduction

This paper is devoted to an investigation of several effects of a magnetic field on stationary flows of an incompressible, electrically conducting fluid. The flow field is assumed due to the uniform motion of a finite obstacle in an unbounded region. An approximate description of the pressure and velocity is given, assuming that (i) the magnetic field is uniform and parallel to the velocity at infinity, and (ii) the magnetic field is strong. The precise meaning of (ii) in the present problem will be clear from the results. For simplicity, the discussion is restricted to the two-dimensional case. However, no new methods appear to be required for analogous problems with axial symmetry.

A perturbation method is used, based upon the assumption that the Hartmann number M is large compared to unity. The limit process actually considered rests upon the additional stipulations

$$\lim_{M\to\infty} (Re/M) = 0, \tag{1}$$

$$\lim_{M\to\infty} (Rm/M) = 0, \tag{2}$$

where Re and Rm are, respectively a Reynolds number and a magnetic Reynolds number of the flow, both regarded here as definite functions of M. We note that (1) and (2), together with the additional requirement that the magnetic field be unperturbed by the presence of the body, in the absence of a flow, appear sufficient to warrant the assumption (i) above. This point will not be discussed in detail here; however, additional comment may be found in Appendix A.

The problem posed above has been discussed previously by Chester (1961), and Chester & Moore (1961). The approximate solution given by Chester is derived from the solution of the linear (Stokes) equations, obtained by neglecting entirely inertial effects, but retaining the magnetohydrodynamic body force. Conditions (1) and (2) then emerge from the requirement that this solution satisfy also, approximately, the full Navier–Stokes equations. The justification for this procedure is based largely upon the global behaviour of the solution as $M \to \infty$, and, as was pointed out by Chester & Moore, requires in particular a careful analysis of the orders of magnitude near the point at infinity.

In the present paper the results of Chester are rederived for the two-dimensional flow by a method which introduces explicitly a limit valid near the point at infinity. It is shown that an approximation to velocity and pressure which is valid uniformly over the flow field, excluding possibly small neighbourhoods of isolated points, is itself a 'boundary-layer' approximation to a solution of the linear Stokes equations. The desired asymptotic solution is then defined by a boundary-value problem for a simpler, approximate system of partial-differential equations. Although this procedure seems to be conceptually clearer in its treatment of the effect of the strong magnetic field, additional complications occur in the selection of the boundary conditions at the obstacle, and, in so far as the limit is concerned, the perturbation method appears to offer no special advantages. However, the analysis of certain details of the approximation, including the flow field near the boundary and the first correction for the effect of Reynolds number, may be easily included in the present construction, using essentially the same methods as before.

The starting point of the present investigation is the observation that, for sufficiently large M, the velocity normal to any line of force may be made arbitrarily small, *uniformly over the exterior region*. Thus, in the case of two-dimensional flow, the limit obtained over any finite region is essentially one-dimensional. Clearly, however, any solution of this type is incompatible with all boundary conditions (e.g. the velocity vanishes identically on any field line which intersects the boundary) and hence is, in the limit, indeterminate. However, it can be shown that the indeterminacy is only apparent and is due, in part, to the singular nature of the limit at the point at infinity. It will also be shown that a second limit (the outer limit), which is valid in a neighbourhood of infinity, is related in a natural way to an approximation which is valid uniformly over the entire exterior region. For this reason, the asymptotic series which is valid at large distances from the body will be examined in detail (see § 3).

The principal new results of this paper are contained in §§ 3–6. In § 3 the influence of the nominally small non-linear inertial terms (that is, the first correction for the effect of Reynolds number) is estimated and, in part, calculated explicitly. It is found that these corrections are in fact uniformly small, in agreement with the conjecture of Chester, provided (1) holds. However, the error is larger near lines tangent to the boundary than at other finite points. Also, an explicit solution representing a velocity boundary layer on surfaces not penetrated by the magnetic field is given in § 4. The effect of such a boundary layer on the overall drag experienced by the solid is given in § 5.

The methods of the present paper appear to provide a direct approach to the effect of Reynolds number in cases where (1) may not be satisfied. The essential point here is the observation that an application of the previous procedures, assuming now that

$$\lim_{M\to\infty} (M^2/Re^2) = \kappa \quad (0 < \kappa < \infty), \tag{3}$$

leads formally to a one-parameter family of approximations, with parameter κ, which coincides with the former result as $\kappa \to \infty$. The relevance of this result depends on whether or not a boundary-value problem for an approximate, non-linear system is solvable. No attempt is made here to resolve this question. However, assuming tentatively that the necessary limit exists, a generalization of the expansion of the drag can be given.

2. Formulation of the problem

We consider the stationary flow of a viscous, conducting fluid past a finite cylindrical obstacle, the permeability of the latter being equal to that of the fluid. The unperturbed stream has a velocity $U\boldsymbol{i}$, and the flow is subjected to a parallel magnetic field which is uniform at infinity. For simplicity, the magnetic field will be assumed to be unperturbed, i.e. equal to $H\boldsymbol{i}$ everywhere (see Appendix A). The electric field may be assumed to vanish everywhere.†

The exact equations and boundary conditions then may be written in the following dimensionless form.

$$Re\, \boldsymbol{q}.\nabla\boldsymbol{q} + \nabla p + M^2[\boldsymbol{q} - (\boldsymbol{q}.\boldsymbol{i})\,\boldsymbol{i}] - \nabla^2\boldsymbol{q} = 0, \tag{4}$$

$$\nabla.\boldsymbol{q} = 0, \tag{5}$$

$$\boldsymbol{q} = \boldsymbol{i}, \quad p = 0 \quad \text{(at infinity)}, \tag{6}$$

$$\boldsymbol{q} = 0 \quad \text{(at the body)}, \tag{7}$$

where $$Re = UL/\nu, \quad M = \mu HL(\sigma/\rho\nu)^{\frac{1}{2}}$$

and ρ the density, ν the kinematic viscosity, μ the permeability, and σ the conductivity are constants. The physical (primed) variables are

$$\begin{aligned}
\boldsymbol{q}' &= Uu\boldsymbol{i} + Uv\boldsymbol{j} = U\boldsymbol{q}, & \boldsymbol{q}' &= \text{velocity};\\
p' &= (\rho U\nu/L)\,p + p'_\infty, & p' &= \text{pressure};\\
x' &= Lx, \quad y' = Ly, & L &= \text{characteristic body dimension.}
\end{aligned}$$

The origin of the coordinate system and the length L may be chosen so that the two lines (the tangent lines) $-\infty < x < +\infty, y = \pm 1$ pass through the uppermost and lowermost points of the boundary. It is assumed that the cylinder is convex and that $L > 0$. The boundary will consist of a closed curve which is analytic, except possibly at isolated points on the tangent lines.

The formulation may be completed by prescribing the function $Re(M)$. Actually, only three distinct cases appear, depending upon whether M/Re tends to infinity, a finite positive number, or zero, as M tends to infinity. The greater

† The two-dimensional problem is here taken as the limit of a suitably chosen, axially symmetric problem, as a characteristic mean curvature tends to zero.

part of this paper is devoted to the first case, although we shall comment briefly on the more general case in § 6. In order to bring out at an early stage the non-linear aspect of the problem, we shall assume that $Re = O(M^{\frac{1}{2}})$ and define

$$Re = R\epsilon^{-\frac{1}{2}} \quad (\epsilon = M^{-1}), \tag{8}$$

where R is a fixed, positive number, and ϵ is regarded as a small parameter.

We shall describe a procedure for constructing a uniformly valid approximation to the solution (or solutions) of the exact problem. The sense in which the approximation is valid will be clear from its construction; however, some indication of the form of the asymptotic representation emerges from the following argument. Suppose that we wish to determine the behaviour of p and $\boldsymbol{q}$ at a *finite* point. Defining $p^* = \epsilon^2 p$, and formally passing to the limit $\epsilon = 0$, the exact equations (4), (5) reduce to

$$\nabla p^* + [\boldsymbol{q} - (\boldsymbol{q}.\boldsymbol{i})\,\boldsymbol{i}] = 0, \tag{9a}$$

$$\nabla.\boldsymbol{q} = 0, \tag{9b}$$

a general solution of which is

$$\boldsymbol{q} = [xf_1''(y) + f_2(y)]\,\boldsymbol{i} - f_1'(y)\,\boldsymbol{i}, \tag{10a}$$

$$p^* = f_1(y), \tag{10b}$$

where f_1 and f_2 are arbitrary functions and primes denote differentiation with respect to y. The following facts are immediately apparent. First, there is no solution of the form (10a) which satisfies all of conditions (6), (7). Secondly, if conditions at infinity are relaxed, the limit is not uniquely determined by the remaining constraints. This suggests the existence of an outer limit (Kaplun & Lagerstrom 1957; Lagerstrom & Cole 1955) which describes the formation and decay of disturbances near the point at infinity. The essentially one-dimensional limit (10)† expresses the *regularity* of the outer solution at all finite points $(x, y \neq \pm 1)$. It is not surprising, therefore, that the central issue in the present problem is the construction of an approximation which is valid at large distances from the body.

3. The flow field at large distances

We propose to study the structure of the flow field at points which lie on field lines situated a finite distance from the origin, but such that $|x| = O(\epsilon^{-1})$. Accordingly, we introduce the new variables

$$\tilde{x} = \epsilon x, \quad \tilde{y} = y, \tag{11a}$$

$$\tilde{u} = u, \quad \tilde{v} = \epsilon^{-1}v, \quad \tilde{p} = \epsilon p. \tag{11b}$$

The terms of an (outer) expansion of $\tilde{u}$, $\tilde{v}$, and $\tilde{p}$, valid for $\tilde{x}$ and $\tilde{y}$ fixed, up to and including those of order $\epsilon^{\frac{1}{2}}$, will be exhibited explicitly. This outer expansion has the form

$$\tilde{F} = \sum_{i=0}^{n} \delta_i(\epsilon)\, F_i(\tilde{x}, \tilde{y}) + o(\delta_n), \tag{12}$$

† It will appear that $f_1(y)$ is piecewise constant.

where $\delta_i(\epsilon)$, $i = 0, 1, \ldots$, is a sequence of functions satisfying

$$\delta_0 = 1, \quad \delta_1 = \epsilon^{\frac{1}{2}}, \quad \lim_{\epsilon \to 0} (\delta_{i+1}/\delta_i) = 0 \quad (i = 1, 2, \ldots) \tag{13}$$

and F denotes u, v, or p. By insertion of the outer expansion into the exact equations (written in the appropriate variables), one obtains the following equations for the terms in question:

$$\partial p_0/\partial \tilde{x} - \partial^2 u_0/\partial \tilde{y}^2 = 0, \tag{14a}$$

$$\partial p_0/\partial \tilde{y} + v_0 = 0. \tag{14b}$$

$$\partial u_0/\partial \tilde{x} + \partial v_0/\partial \tilde{y} = 0, \tag{14c}$$

$$\partial p_1/\partial \tilde{x} - \partial^2 u_1/\partial \tilde{y}^2 = -u_0\, \partial u_0/\partial \tilde{x} - v_0\, \partial u_0/\partial \tilde{y}, \tag{15a}$$

$$\partial p_1/\partial \tilde{y} + v_1 = 0, \tag{15b}$$

$$\partial u_1/\partial \tilde{x} + \partial v_1/\partial \tilde{y} = 0. \tag{15c}$$

If $\delta_{\nu+1} = \epsilon$, succeeding terms to order $\delta_\nu(\epsilon)$ inclusive again satisfy the homogeneous system (14). To obtain the governing conditions on the terms of (12), one notes that as $\epsilon \to 0$, $\tilde{x}$, $\tilde{y}$ fixed, the boundary tends to the segment $\tilde{x} = 0$, $-1 \leqslant \tilde{y} \leqslant 1$. The initial conditions there to $O(1)$ are supplied by (6), provided the end-points are excluded. Additional conditions are contained in the following remark: any partial sum of (12) is bounded, uniformly over any domain of the type $|\tilde{x}| \geqslant A\epsilon^\alpha$, $-\infty \leqslant \tilde{y} \leqslant +\infty$ (denoted below by D_α), as $\epsilon \to 0$, where A is an arbitrary positive constant and $0 \leqslant \alpha < 1$. The condition of boundedness over a domain D_α is used here as a governing condition on the series (12). However, as will be clear later, boundedness in this sense is a direct consequence of matching conditions at the $\tilde{y}$-axis, which are associated with the individual terms of (12) (cf. Kaplun & Lagerstrom 1957). In fact, our assertion may be justified by showing that the presence of boundary layers does not introduce unwanted orders into (12). In this respect, additional matching conditions are generally required to render the series unique (cf. §4).

We shall further require that

$$u_0 = 1, \quad v_0 = p_0 = 0; \quad u_i = v_i = p_i = 0 \quad (i = 1, 2, \ldots) \text{ at infinity}, \tag{16}$$

$$u_i,\ v_i,\ p_i, \quad \text{regular on} \quad \tilde{x} = 0, \quad |\tilde{y}| > 1 \quad (i = 0, 1, 2, \ldots). \tag{17}$$

Conditions (17) state that a series valid for $\tilde{x} > 0$ determines, by analytic continuation across the $\tilde{y}$-axis, a series valid in $\tilde{x} < 0$.

Eliminating v_0 and p_0 from (14), we have

$$\left(\frac{\partial}{\partial \tilde{x}} - \frac{\partial^2}{\partial \tilde{y}^2}\right)\left(\frac{\partial}{\partial \tilde{x}} + \frac{\partial^2}{\partial \tilde{y}^2}\right) u_0 = 0. \tag{18}$$

A solution of (18) satisfying the necessary conditions at $|\tilde{x}| = 0, \infty$ is given by

$$E(\tilde{x}, \tilde{y}) = 1 - \frac{1}{\sqrt{\pi}} \int_r^s e^{-t^2}\, dt, \quad r = \frac{\tilde{y} - 1}{2\, |\tilde{x}|^{\frac{1}{2}}}, \quad s = \frac{\tilde{y} + 1}{2\, |\tilde{x}|^{\frac{1}{2}}}. \tag{19}$$

Also,
$$\left(\frac{\partial}{\partial \tilde{x}}+\frac{\partial^2}{\partial \tilde{y}^2}\right)(p_0-u_0) = 0, \tag{20a}$$
$$\left(\frac{\partial}{\partial \tilde{x}}-\frac{\partial^2}{\partial \tilde{y}^2}\right)(p_0+u_0) = 0. \tag{20b}$$

Now the initial value problem for (20a) has, in general, no bounded solutions in the right half-plane (Petrovsky 1954). The only acceptable solution is
$$p_0-u_0 = -1 \quad (\tilde{x} > 0). \tag{21a}$$
Similarly, in the left half-plane,
$$p_0+u_0 = 1 \quad (\tilde{x} < 0). \tag{21b}$$
Hence, if p_0 is regular on $\tilde{x} = 0$, $|\tilde{y}| > 1$, then $u_0(0,\tilde{y}) = 1$ (and also $p_0(0,\tilde{y}) = 0$) for $|\tilde{y}| > 1$. Consequently u_0 can differ from (19) by at most a homogeneous solution (eigensolution) of (18), which vanishes at all points of the $\tilde{y}$-axis excluding $\tilde{y} = \pm 1$. The eigensolution is then eliminated by requiring that the limit of u_0 as $\tilde{x} \to 0$ exist at the exceptional points, this being an immediate consequence of boundedness over D_α.

The solution defined by (14b), (19) and (21) coincides over any finite region with a discontinuous flow field. At the tangent lines, shear layers actually occur, across which, by the action of viscous and magnetic stresses, the pressure and velocity undergo a rapid transition. The appearance of these shear layers was pointed out previously by Chester (1961).

The terms of order $\epsilon^{\frac{1}{2}}$ may be obtained in an analogous fashion. The relations
$$\left(\frac{\partial}{\partial \tilde{x}}-\frac{\partial^2}{\partial \tilde{y}^2}\right)(p_1+u_1) = \left(\frac{\partial}{\partial \tilde{x}}+\frac{\partial^2}{\partial \tilde{y}^2}\right)(p_1-u_1) = -u_0\frac{\partial u_0}{\partial \tilde{x}}-v_0\frac{\partial u_0}{\partial \tilde{y}}, \tag{22}$$
which follow directly from (15), are to be integrated. The procedure is essentially as before, and we shall note here only that the eigensolution of order $\epsilon^{\frac{1}{2}}$ is nontrivial; any linear combination of the functions $S(\tilde{x},\tilde{y}+1)$, $S(\tilde{x},\tilde{y}-1)$ may be included in u_1, where
$$S(x,y) = -\frac{1}{4}\frac{\exp(-y^2/4|x|)}{(\pi|x|)^{\frac{1}{2}}} \tag{23}$$
is the fundamental solution of the homogeneous equation (18). This follows from the fact that $\epsilon^{\frac{1}{2}}S(\tilde{x},\tilde{y})$ is $O(\epsilon^{1-\frac{1}{2}\alpha})$ uniformly over D_α.

Other properties of the terms u_1, v_1, p_1 which will be of interest follow from the expressions given in Appendix B.

The partial sum of (12) of order $\epsilon^{\frac{1}{2}}$ inclusive gives, when evaluated for small $\tilde{x}$, the following expansion valid at finite exterior points $(x, y \neq \pm 1)$:
$$u = \begin{cases}(Re/M^2)\,[xk''(y)+l\pm(y)]+\ldots & (|y|<1),\\ 1+(Re/M^2)\,xk''(y)+\ldots & (|y|>1),\end{cases} \tag{24a}$$
$$p = \begin{cases}-M\,\mathrm{sgn}\,(x/\epsilon)+Re\,k(y)+\ldots & (|y|<1),\\ Re\,k(y)+\frac{1}{4}+\ldots & (|y|>1),\end{cases} \tag{24b}$$
where
$$k(y) = \frac{\sqrt{3}}{4\pi}\log\left(\frac{|y^2-1|}{y^2+3}\right), \tag{24c}$$

and $l \pm (y)$ are functions, generally different in the upstream and downstream regions, which are determined by requiring that the tangency condition at the boundary be satisfied to order Re/M^2 inclusive. This means that all terms of the outer expansion of u to order $\epsilon^{\frac{3}{2}}$ exclusive must vanish on $\tilde{x} = 0$, $|\tilde{y}| < 1$.

Two interesting conclusions can be drawn from (24). In the first place, the tangent lines, which, to the first approximation, consist of those points where the limit (10) is discontinuous, become, in the second approximation, the locus of singularities of p. As far as the asymptotic representation for small Re/M is concerned, the tangent lines are excluded anyway, and the singularities appear to be harmless.† However, the state of affairs if Re/M is not small may not be simple (see § 6). In the second place, it is clear from (24a) that the vorticity in the region $|y| > 1$ decays algebraically near infinity. Now algebraic decay of vorticity is not observed in conventional viscous flows, and hence must here be due entirely to the presence of a magnetic field. Intuitively, the reason for the algebraic term is clear from the manner in which (22) is solved. The fundamental solution of the homogeneous equation differs from that of the heat equation by its 'upstream influence'. Consequently, the effect of a source distribution over any finite region (the non-linear effect) need not approach a limit exponentially as $\tilde{x} \to 0$, even though the governing conditions on the $\tilde{y}$-axis are satisfied to the requisite order.‡ At the same time, the appearance of algebraic terms as a non-linear effect provides a check on the correctness of (24) for all functions $Re(M)$ satisfying (1).

4. Boundary layers

It was noted above that the boundedness conditions rest upon a knowledge of certain terms which are introduced in the outer expansion by the presence of boundary layers. Here, it will be shown that the effect of boundary layers is of order unity over any finite region, excluding exceptional points on the tangent lines. These remarks appear to justify the application of the boundedness condition to all orders and also serve as a basis for computing successively higher-order boundary-layer corrections.

A basic result is that the outer expansion, which is derived with the intent of describing the flow field at large distances, is actually valid over a large region, and, under certain conditions, may be valid over some *finite* region. Consider, for example, the limits u_0, v_0, p_0. Expressed as functions of x, y these terms evidently provide an approximation to order unity, valid uniformly over space (excluding isolated points) if and only if (i) the boundary is strictly convex at the tangent lines and (ii) the uppermost and lowermost points of the boundary lie (to order unity) on the y-axis. If (i) does not hold, the effect of a boundary layer (the longitudinal boundary layer) is of order unity on the tangent lines. If (ii) is relaxed, a uniformly valid approximation may still be obtained by appropriate

† Closer examination of the expansion of $\tilde{p}$ shows that for x fixed, $y \to \pm 1$, there is a term of order $\epsilon^{\frac{1}{2}} \log \epsilon$.

‡ Since essentially the same argument could be advanced whenever infinitesimal perturbations generate a forward-running wake, it appears that vorticity and current decay algebraically whenever $(M^2/Rm\,Re) > 1$.

shifts of the origins of shear layers. However, if errors are to be smaller than $\epsilon^{\frac{3}{4}}$, the shape of the boundary in the strip $|y| < 1$, and the presence there of a (transverse) boundary layer, must be considered (cf. remark following equation (24)). We shall examine these problems in turn.

Consider first the longitudinal boundary layer. Since the boundary is convex, we may assume that that portion of the contour lying on the tangent lines consists of the two segments $-\lambda_1 \leqslant x \leqslant +\lambda_1$, $y = +1$, and $-\lambda_2 \leqslant x+x_0 \leqslant +\lambda_2$, $y = -1$, x_0 being a constant, $\lambda_1, \lambda_2 \geqslant 0$. The shear layers now originate from the end-points of these segments. If λ_1 or λ_2 is positive, a boundary layer of thickness $O(\epsilon^{\frac{1}{2}})$ is developed. We wish to show that a solution valid in the boundary layer uniquely determines the outer expansion to order $\epsilon^{\frac{1}{2}}$ inclusive. Consider the upper segment. The boundary-layer equations are obtained from (4), (5) by passage to the limit for $\epsilon = 0$ in the variables

$$x, \quad \bar{y} = \epsilon^{-\frac{1}{2}}(y-1), \quad \bar{u} = u-1, \quad \bar{v} = \epsilon^{-\frac{1}{2}}v, \quad \bar{p} = \epsilon p.$$

The result is formally the same as (14) and hence

$$\left(\frac{\partial}{\partial x}-\frac{\partial^2}{\partial \bar{y}^2}\right)\left(\frac{\partial}{\partial x}+\frac{\partial^2}{\partial \bar{y}^2}\right)\bar{u} = 0, \tag{25a}$$

$$\bar{u} = -1 \quad \text{when} \quad \bar{y} = 0, \tag{25b}$$

$$\bar{u} = 0 \quad \text{when} \quad \bar{y} = \infty, \tag{25c}$$

in the interval $-\lambda_1 \leqslant x \leqslant +\lambda_1$. In the limit, the body occupies the semi-infinite strip $-\lambda_1 \leqslant x \leqslant +\lambda$, $\bar{y} \leqslant 0$ in the $(x,\bar{y})$-plane. Hence, for $|x| \geqslant \lambda_1$, we require also that $\bar{u} = 0$ at infinity and $\bar{u} = -1$ on $|x| = \lambda_1, \bar{y} < 0$.

The boundary-value problem (25) may be solved by distributing sources on $\bar{y} = 0$. Thus

$$\bar{u}(x,\bar{y}) = \int_{-\lambda_1}^{+\lambda_1} K(\xi)\, S(x-\xi,\bar{y})\, d\xi, \tag{26}$$

where $S(x,y)$ is given by (23). Condition (25b) is satisfied provided that

$$\frac{1}{4\sqrt{\pi}}\int_{-\lambda_1}^{+\lambda_1} \frac{K(\xi)}{|x-\xi|^{\frac{1}{2}}}\, d\xi = 1.$$

This integral equation has been studied by Carleman (1922). The solution is

$$K(x) = (8/\pi)^{\frac{1}{2}}\,(\lambda_1^2-x^2)^{-\frac{1}{4}},$$

which immediately yields the distribution of skin friction on $\bar{y} = 0$. The distribution of pressure is given by

$$(\bar{p})_{\bar{y}=0} = \int_{-\lambda_1}^{+\lambda_1} K(\xi)\,\operatorname{sgn}(x-\xi)\, S(x-\xi,0)\, d\xi = \Pi(-x)-\Pi(x),$$

where $$\Pi(x) = 4\sqrt{\pi}\,\Gamma^{-2}(\tfrac{1}{4})\left(\frac{\lambda_1+x}{2\lambda_1}\right)^{\frac{1}{4}} F\left(\frac{1}{4},\frac{3}{4};\frac{5}{4};\frac{\lambda_1+x}{2\lambda_1}\right).$$

Evidently, the surface pressure distribution caused by the boundary layer will contribute to the moment but not to the lift.

The solution in $|x| > \lambda_1$ is obtained by superimposing on (26) a distribution of 'upstream' or 'downstream' sources (i.e. fundamental solutions of the heat equation or its adjoint) on the vertical portions of the deformed boundary. The calculation is straightforward and the result need not be given here. The boundary layer on the lower segment may be derived from the above by a simple change of variables.

The constants a and b occurring in the terms u_1, v_1, p_1 of the outer expansion (see Appendix B) are equal, respectively, to the integrated strength of the source distribution representing the upper and lower boundary layer. The precise matching condition states that the partial sum of (12) to order $\epsilon^{\frac{1}{2}}$ inclusive is valid to $O(\epsilon^{1-\frac{1}{2}\alpha})$ uniformly over D_α if and only if

$$a = \frac{1}{2}\int_{-\lambda_1}^{+\lambda_1} K(\xi)\,d\xi = c\lambda_1^{\frac{1}{2}}, \quad c = \frac{2^{\frac{1}{2}}\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})}, \quad b = c\lambda_2^{\frac{1}{2}}.$$

By introducing additional limits, valid at the end-points $(x,y) = (\lambda_1, 1)$, etc., the orders $\delta_i(\epsilon)$ $(i = 2, 3, \ldots, \nu)$ and the corresponding terms of (12) may be determined. The nature of the problem is illustrated by the following example. We assume that $\lambda_1 = 0$ and that the boundary has a finite radius of curvature at the point $(x,y) = (0,1)$. The equations satisfied by the approximation valid near this point may be obtained in the usual way, by passing to the variables $x^* = \epsilon^{-\frac{1}{3}}x$, $y^* = \epsilon^{-\frac{2}{3}}(y-1)$, u, $\epsilon^{-\frac{1}{3}}v$, ϵp. Again we obtain essentially (25a), but (25b) is replaced by $u = 0$ on $y^* = -\frac{1}{2}\lambda(x^*)^2$; the problem thus involves a moving boundary. Proceeding as before, one concludes that, for this geometry, an eigensolution of order $\epsilon^{\frac{2}{3}}$ is contained in (12), and that the eigensolution consists of a source placed at the origin of the upper shear layers.

The transverse boundary layer is distinguished from the limit just described by the fact that it occurs only over those portions of the boundary penetrated by the magnetic field (cf. Stewartson 1960). In the present problem, this layer is required in order to satisfy (7) to order $\epsilon^{\frac{3}{2}}$ inclusive. Consider, for example, the case of a vertical flat plate. For this boundary, an expansion valid for $|x| \geqslant 0$, $|y| < 1$ is given by

$$u = R\epsilon^{\frac{3}{2}}xk''(y) + \ldots, \tag{27a}$$

$$v = R\epsilon^{\frac{3}{2}}(e^{-|x|/\epsilon} - 1)\,k'(y) + \ldots, \tag{27b}$$

$$\tilde{p} = -\operatorname{sgn} x + \epsilon^{\frac{1}{2}}R[k(y) + \tfrac{1}{4}] + \ldots. \tag{27c}$$

In (27b) we have included a term due to the transverse layer. The construction of this term is obvious from the manner in which it appears.

5. Force and moment

The leading term of the outer expansion is sufficient to determine the contributions to the dimensionless lift and moment which are of order M, at least for the class of bodies considered here. If the body is strictly convex at the tangent lines, only the uniform pressure $+M$ on upstream parts and $-M$ on downstream parts need be considered. The expansion of the drag is found to be

$$D = D'/\rho U\nu = 4M + (c\lambda_1^{\frac{1}{2}} + c\lambda_2^{\frac{1}{2}})\,M^{\frac{1}{2}} + o(M^{\frac{1}{2}}), \tag{28}$$

where D' is the drag in physical units, and λ_1 and λ_2 are defined in § 4. If the first term of this expansion is omitted and $\lambda_1 = \lambda_2 = 1$, the result gives the first approximation to the drag of a horizontal flat plate of length L. Except for this special case, (28) is intended to apply only if the area enclosed by the boundary is of order L^2.

The form of the outer expansion indicates that the first contribution to the right-hand side of (28), arising from the non-linear terms, is of order Re^2/M. Consequently, if (2) is replaced by the linear (Stokes) equations, and the expansion of the drag calculated to order $\delta(M)$ inclusive, the result will agree numerically with the Navier–Stokes value if $M\,\delta(M) \gg Re^2$. The last condition is the proper extension of (1) to higher-order computations of the drag, starting from the linear equations.

6. A generalized limit

The significance of the second approximation to the flow field at large distances is emphasized by the following remark: given a convex, analytic boundary, the second term of the outer expansion becomes arbitrarily large as $\tilde{x} \to 0$, $\tilde{y} = \pm 1$. This follows from the fact that the pressure term contains a factor $\log \tilde{x}$ when expanded near the points in question. Consequently, assumption (1) is evidently necessary if the series is to have any meaning. However, the relation between our series and approximations valid for *Re/M fixed* is of interest and will be commented upon here.

Now the approximation derived above may be termed a 'Stokes' approximation in that linearization in the usual way is possible. For Re/M fixed, $M \to \infty$, it appears that the quadratic inertial terms must be partially accounted for in the first approximation. Let us replace the parameters M and Re by

$$N = M^2/Re, \quad \kappa = M^2/Re^2,$$

and consider approximations valid as $N \to \infty$, assuming now that (3) holds. The outer variables (11) are to be replaced by

$$\tilde{u} = u, \quad \tilde{v} = Nv, \quad \tilde{p} = p/Re; \quad \tilde{x} = x/N, \quad \tilde{y} = y.$$

In place of u_0, v_0, p_0 we consider now the limits u', v', p', satisfying

$$u'\frac{\partial u'}{\partial \tilde{x}} + v'\frac{\partial u'}{\partial \tilde{y}} + \frac{\partial p'}{\partial \tilde{x}} - \kappa\frac{\partial^2 u'}{\partial \tilde{y}^2} = 0, \tag{29a}$$

$$\frac{\partial p'}{\partial \tilde{y}} + v' = 0, \tag{29b}$$

$$\frac{\partial u'}{\partial \tilde{x}} + \frac{\partial v'}{\partial \tilde{y}} = 0. \tag{29c}$$

It is clear that, as stated above, it is not now possible to follow the sequence of steps used in the linear problem, without considering at the outset the effect of the quadratic acceleration terms on the transfer of momentum in the direction of the field. Essentially the same conclusion may be drawn concerning the longitudinal boundary layer. (This layer now is of order $N^{-\frac{1}{2}}$ in thickness.)

From the point of view of the present paper, the system (29) is of interest provided that there exist, for each positive κ, solutions which satisfy all boundary and matching conditions imposed upon the outer limit. If this is the case, it is plausible that there also exist a one-parameter family of solutions, depending upon κ, which tends to the outer limit considered previously, as $\kappa \to \infty$.

If it is assumed that the needed solutions exist, however, the subsequent dependence of approximations upon the parameters leads to results which may be tested in the laboratory. We give one such consequence. Consider the extension of the drag formula (28). One finds for $N \to \infty$, κ fixed and positive,

$$D'/\rho U^2 L = 4K_0(\kappa) + (c\lambda_1^{\frac{1}{2}} + c\lambda_2^{\frac{1}{2}})\, K_1(\kappa)\, N^{-\frac{1}{2}} + o(N^{-\frac{1}{2}}), \tag{30}$$

where λ_1 and λ_2 are defined in § 4, and K_0 and K_1 are unspecified functions which are universal for the class of bodies considered. If (28) and (30) are to agree in the limit of large κ, then

$$K_0(\kappa) \sim \kappa^{\frac{1}{2}}, \quad K_1(\kappa) \sim \kappa^{\frac{3}{4}} \quad \text{as} \quad \kappa \to \infty. \tag{31}$$

The conditions under which (30) and (31) are intended to hold remain as before.

7. Concluding remarks

The practical application of the theory of § 3 is very likely to be limited by the requirement that M/Re be large compared to unity,† and it is therefore desirable to extract as much information as is possible concerning the effect of Re from the linear problem. The basic result of § 3 is that the outer limit introduces, by an interaction process arising from the acceleration of the fluid along the field lines, singularities in the limit attained over a finite region. It is at least suggested that the flow in the shear layers may be altered significantly (e.g. by the rapid rise of the maximum velocity attained in the layer) as M/Re decreases from a large value. Moreover, it is conceivable that the limit proposed in § 6 does not remain bounded as $|\tilde{x}| \to 0$ in the tangent lines.

If (29a) is linearized in the manner of Oseen (the acceleration terms being replaced by $\partial \tilde{u}/\partial \tilde{x}$), it seems likely that the system should provide a better qualitative picture of the effect of Reynolds number than can be obtained by expanding for large κ. For example, the first approximation to u is not symmetric in $\tilde{x}$. It can also be shown that the perturbation on this linear problem leads again to singularities (in both u and p) on the tangent lines. It would appear, therefore, that the singularities are not artificial, that is, cannot be eliminated by partially accounting for the inertia of the fluid through the linear term.

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† The author is indebted to Dr T. Maxworthy for bringing this point to his attention For example, in liquid sodium, with $H = 10^4$ G, $U = 10^{-1}$ m/sec, M/Re is approximately 1.

Appendix A. Perturbations in the magnetic field

Under the assumptions of § 2, the dimensionless magnetic field components $h_1 = h_1'/H$, $h_2 = h_2'/H$ satisfy

$$\frac{\partial h_1}{\partial y} - \frac{\partial h_2}{\partial x} = Rm(vh_1 - uh_2), \tag{A 1}$$

where $Rm = \sigma\mu UL$ = magnetic Reynolds number. Expressed in the variables (11), and with $\tilde{h}_1 = h_1$, $\tilde{h}_2 = Mh_2$, (A 1) reads

$$\frac{\partial \tilde{h}_1}{\partial \tilde{y}} = \frac{Rm}{M}(\tilde{v}\tilde{h}_1 - \tilde{u}\tilde{h}_2) + \frac{1}{M^2}\frac{\partial \tilde{h}_2}{\partial \tilde{x}}. \tag{A 2}$$

Since $\tilde{h}_1$ and u are, by virtue of the exact boundary conditions, of order unity in a neighbourhood of infinity, (A 2) implies that the perturbation of the uniform magnetic field is of order Rm/M there. If (2) is adopted as a condition on Rm, it then follows that the magnetic field tends to $\boldsymbol{i}$ uniformly over D_α. If, in addition, $h = \boldsymbol{i}$ exactly when there is no flow, then the same statement holds when a flow is present, provided isolated points on the tangent lines are excluded. Writing

$$\tilde{h}_1 = 1 + \eta h_1^{(1)}(\tilde{x}, \tilde{y}) + o(\eta), \quad \eta = Rm/M, \tag{A 3}$$

$$\tilde{h}_2 = \eta h_2^{(2)}(\tilde{x}, \tilde{y}) + o(\eta), \tag{A 4}$$

and inserting these expansions, along with (12), into (A 2) and the continuity equation, perturbations in the magnetic field satisfy

$$\partial h_1^{(1)}/\partial \tilde{y} = v_0, \quad \partial h_1^{(1)}/\partial \tilde{x} + \partial h_2^{(1)}/\partial \tilde{y} = 0,$$

$$h_1^{(1)} = h_2^{(1)} = 0 \quad \text{(at infinity)}.$$

Thus
$$h_1^{(1)} = -p_0, \quad h_2^{(1)} = \partial u_0/\partial \tilde{y}.$$

This solution is bounded over D_α by virtue of the condition on the first-order terms in the outer expansion.

Note that the effect of conditions on the magnetic field at the surface of a finite conductor will introduce harmonic perturbations in the magnetic field which are $O(\eta)$ in any finite region. However, this part of the perturbation is $o(\eta)$ under the outer limit process and does not enter into (A 3), (A 4). The approximation to magnetic field which is valid uniformly over the entire exterior region is, therefore, of the form

$$\boldsymbol{h} \sim \boldsymbol{i} + \eta \boldsymbol{h}^{(1)}(x/M, y) + \eta \boldsymbol{h}^{(2)}(x, y),$$

$$\nabla \times \boldsymbol{h}^{(2)} = \nabla . \boldsymbol{h}^{(2)} = 0. \tag{A 5}$$

The solution (A 5) differs from that given by Chester (1961) only by terms which are $o(\eta)$ uniformly over the exterior region.

The above arguments may be applied to the limits considered in § 6, with the result that (2) is replaced by

$$\lim_{N\to\infty} Rm(N)/N = 0. \tag{A 6}$$

We note that (2) implies (A 3) whenever the parameter κ is fixed and positive.

Appendix B. The second approximation to velocity and pressure

The terms u_1, v_1, and p_1 of the outer expansion may be found by a straightforward calculation and are given by

$$u_1 = R\left[\tfrac{1}{4}\psi_0\frac{\partial u_0}{\partial \tilde{y}} - \tfrac{1}{8}u_0^2 + \frac{\sqrt{3}}{16\pi}(\phi_1-\phi_2)\right]\operatorname{sgn}\tilde{x} + \phi_3,$$

$$p_1 = R\left[\tfrac{1}{4}\psi_0\frac{\partial u_0}{\partial \tilde{y}} - \tfrac{3}{8}u_0^2 + \frac{\sqrt{3}}{16\pi}(\phi_1+\phi_2) + \tfrac{1}{4}\right] + \phi_3\operatorname{sgn}\tilde{x},$$

$$v_1 = -\frac{\partial p_1}{\partial \tilde{y}},$$

where ψ_0 is the stream function for u_0, v_0, defined by

$$\partial\psi_0/\partial\tilde{y} = u_0, \quad \psi_0 = 0 \quad \text{on the } \tilde{x}\text{-axis},$$

and ϕ_1, ϕ_2, and ϕ_3 are defined as follows:

$$\phi_i = g_i(r) + g_i(s) - 2G_i(r,s) \quad (i = 1, 2),$$

$$g_1(z) = 4\int_0^z e^{-t^2}\int_0^t e^{\tau^2}\,d\tau\,dt = \int_0^1 \frac{1-e^{-z^2t}}{t(1-t)^{\frac{1}{2}}}\,dt,$$

$$g_2(z) = 4\int_0^z e^{+t^2}\int_0^t (\sqrt{3}\,e^{-3\tau^2} - e^{-\tau^2})\,d\tau\,dt = \int_0^2 \frac{(1-e^{-z^2t})}{t(1+t)^{\frac{1}{2}}}\,dt,$$

$$G_1(r,s) = \int_0^1 \frac{1-\exp-\frac{1}{4}[(r+s)^2t + 3(r-s)^2t(1-t)^{-1}]}{t(1-t)^{\frac{1}{2}}}\,dt,$$

$$G_2(r,s) = \int_0^2 \frac{1-\exp-\frac{1}{4}[(r+s)^2t + 3(r-s)^2t(1+t)^{-1}]}{t(1+t)^{\frac{1}{2}}}\,dt,$$

$$\phi_3 = aS(\tilde{x},\tilde{y}-1) + bS(\tilde{x},\tilde{y}+1), \quad a, b = \text{const.}$$

The integral representations of g_i and G_i may be evaluated asymptotically to obtain the following expansions for small $\tilde{x}$ (cf. equation (24)):

$$u_1 = R\tilde{x}k''(\tilde{y}) + O(\tilde{x}^3),$$

$$p_1 = R\begin{cases} k(\tilde{y}) + O(\tilde{x}^2) & (|\tilde{y}| > 1), \\ k(\tilde{y}) + \tfrac{1}{4} + O(\tilde{x}^2) & (|\tilde{y}| < 1), \end{cases}$$

where
$$k(\tilde{y}) = \frac{\sqrt{3}}{4\pi}\log\left(\frac{|\tilde{y}^2-1|}{\tilde{y}^2+3}\right).$$